

SMALL NOISE ASYMPTOTICS FOR INVARIANT DENSITIES FOR A CLASS OF DIFFUSIONS: A CONTROL THEORETIC VIEW

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Abstract: We consider multidimensional nondegenerate diffusions with invariant densities, with the diffusion matrix scaled by a small $\epsilon > 0$. The o.d.e. limit corresponding to $\epsilon = 0$ is assumed to have the origin as its unique globally asymptotically stable equilibrium. Using control theoretic methods, we show that in the $\epsilon \downarrow 0$ limit, the invariant density has the form $\approx \exp(-W(x)/\epsilon^2)$, where the W is characterized as the optimal cost of a deterministic control problem. This generalizes an earlier work of Sheu. Extension to multiple equilibria is also given.

Keywords: diffusions, invariant density, small noise limit, Hamilton-Jacobi equation, viscosity solution.

1 Introduction

A recurrent theme in applied mathematics is the resolution of non-uniqueness issue of a deterministic system by considering its perturbation with small noise and recovering a hopefully unique choice by passing to the vanishing noise limit. (This idea is attributed to Kolmogorov in [7], p. 626) To mention two such instances, see the analysis of jump phenomena in nonlinear circuits in [20] and equilibrium selection in evolutionary games in [10]. In fact the notion of ‘viscosity solutions’ we use later in this work can be motivated along these lines.

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A similar situation also arises in the passage from quantum to classical mechanics (the correspondence principle) when one uses the Feynman-Kac device to convert Feynman path integrals to Wiener space integrals by complexifying time, whence the problem reduces to that of small noise behavior of diffusions. Motivated by this, Freidlin and Wentzell have extensively developed the small noise limit theory for diffusions [11].

Building on the classic work of Freidlin and Wentzell [11], Sheu in [21] characterized the small noise limit of the invariant density of a positive recurrent diffusion for a restricted class of diffusions. This was further extended by Day [6]. Our aim is to establish a similar result in a more general set-up, using a novel control theoretic approach based on the theory of viscosity solutions. We do so by exploiting the fact that the ‘adjoint’ partial differential equation satisfied by the invariant density gets converted to the Hamilton-Jacobi-Bellman equation of an associated ergodic control problem via the logarithmic transformation. The limiting case thereof as the noise decreases to zero can then be handled by exploiting the machinery of viscosity solutions.

The paper is organized as follows: The next section introduces the problem and states the main result. Section 3 uses the equivalent control formulation in order to go to the small noise limit and obtain the asymptotic expression for the invariant density as $\approx \exp(-W(x)/\epsilon^2)$. Section 4 obtains a representation for W in terms of a deterministic control problem. Section 5 extends the results to the case of multiple equilibria.

See also [1], [2], [19], [23] for related work.

2 The small noise asymptotics

We consider an d -dimensional diffusion X given by the stochastic differential equation

$$dX(t) = b(X(t))dt + \epsilon\sigma(X(t))dB(t), \quad t \geq 0, \quad (1)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d, \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ are smooth, bounded, with bounded derivatives, and B is an m -dimensional Brownian motion. We assume:

1. $\|D^\kappa b(x)\|, \|D^\kappa \sigma(x)\| \leq C < \infty$ for some C and all x and all multi-indices κ .
2. $a(x) = [[a_{ij}(x)]] \stackrel{def}{=} \sigma(x)\sigma^T(x)$ satisfies:

$$\lambda\|x\|^2 \leq x^T a(y)x \leq \Lambda\|x\|^2, \quad \forall x, y \in \mathbb{R}^d,$$

for some $0 < \lambda \leq \Lambda$.

3. There exist constants $\alpha > 0, 0 < \beta \leq 1$ such that

$$\limsup_{\|x\| \rightarrow \infty} \left[\alpha \Lambda + \|x\|^{1-\beta} b(x)^T \left(\frac{x}{\|x\|} \right) \right] < 0. \quad (2)$$

4. $b(0) = 0$ and zero is the unique globally asymptotically stable equilibrium point of the o.d.e

$$\dot{x}(t) = b(x(t)). \quad (3)$$

Let

$$\mathcal{L}^\epsilon \stackrel{def}{=} \langle b(\cdot), \nabla \rangle + \frac{\epsilon^2}{2} \text{tr} (a(\cdot) \nabla^2)$$

Theorem 1 : For each $\epsilon \in (0, 1)$, $X(\cdot)$ is positive recurrent and the corresponding (unique) stationary distributions $\{\mu^\epsilon, 0 < \epsilon < 1\}$ are tight. Furthermore, $\lim_{\epsilon \downarrow 0} \mu^\epsilon = \delta_0$, where δ_x for $x \in \mathbb{R}^d$ is the Dirac measure at x .

Proof: For $\mathcal{V}(x) \stackrel{def}{=} \|x\|^2$, direct computation shows that

$$\begin{aligned} \mathcal{L}^\epsilon \mathcal{V}(x) &= \epsilon^2 \sum_i a_{ii}(x) + 2b(x)^T x \\ &\leq d\epsilon^2 \Lambda + 2b(x)^T x \\ &\leq d\epsilon^2 \Lambda - \alpha \Lambda \|x\|^\beta \end{aligned} \quad (4)$$

for $\|x\| \geq$ a sufficiently large $K > 0$. (The last inequality follows by virtue of (2).) In particular,

$$\mathcal{L}^\epsilon \mathcal{V}(x) < -\delta$$

for some $\delta > 0$ and sufficiently large $\|x\|$. But this is the standard stochastic Liapunov condition for positive recurrence of nondegenerate diffusions, implying in particular the existence of μ^ϵ . To see this, first note that a simple application of Dynkin formula in conjunction to the above leads to the conclusion that the mean hitting time of $\{x : \|x\| \leq K\}$ from any point is finite (p. 305, [13]). An invariant probability measure can then be constructed as in [16] (see also [17]). Uniqueness follows from the observation that under the stated hypotheses, $X(t)$ has a density $p(y|x, t) > 0$ for all t and therefore so does $\mu^\epsilon(dx) = \int p(x|t, z) \mu^\epsilon(dz)$. Thus any two invariant measures would have to be mutually absolutely continuous. But by a well known fact from ergodic theory of Markov processes (section XIII.4, [24]), two distinct extremal invariant probability measures of a Markov process must be mutually

singular. This contradiction implies that the invariant probability measure is unique. Also, by Proposition 2.3 of [18], we have

$$\int |\mathcal{L}^\epsilon \mathcal{V}| d\mu^\epsilon \leq K < \infty \quad (5)$$

for some K . By Theorem 9.17 of [8], μ^ϵ is characterized by

$$\int \mathcal{L}^\epsilon f d\mu^\epsilon = 0 \quad (6)$$

for twice continuously differentiable f with bounded first and second derivatives.

From (4), $\mathcal{L}^\epsilon \mathcal{V}(x) \leq -g(x)$ for a g satisfying $\lim_{\|x\| \rightarrow \infty} g(x) = \infty$. Thus for a suitable $K^* > 0$, $\mathcal{L}^\epsilon \mathcal{V}(x) < 0$ for $\|x\| \geq K^*$. Using (5), we then have

$$\int_{\|x\| \geq K^*} g(x) d\mu^\epsilon \leq \int_{\|x\| \leq K^*} |\mathcal{L}^\epsilon \mathcal{V}| d\mu^\epsilon \leq K.$$

Since $\lim_{\|x\| \rightarrow \infty} g(x) = \infty$, this implies tightness of $\{\mu^\epsilon, \epsilon > 0\}$. Now let $\epsilon \downarrow 0$ in (6) to obtain

$$\int \mathcal{L}^0 f d\mu = 0$$

for all f as above and all limit points μ in $\mathcal{P}(\mathbb{R}^d)$ of μ^ϵ as $\epsilon \downarrow 0$. By the criterion (6), μ must be an invariant measure for (3). But under our assumptions on (3), this is possible only for $\mu = \delta_0$. \square

Our aim is to capture the precise manner in which $\mu^\epsilon \rightarrow \delta_0$. Note that for $\epsilon > 0$, μ^ϵ will have a density φ^ϵ . First we will prove the following result

Theorem 2 : $\lim_{\epsilon \downarrow 0} \epsilon^2 \ln(\varphi^\epsilon(x)) = -W(x)$ for

$$W(x) \stackrel{def}{=} \inf \left(\int_0^\infty u(t)^T a^{-1}(y(t)) u(t) dt \right), \quad (7)$$

where the infimum is over all measurable $u(\cdot)$ such that the trajectory $y(\cdot)$ of

$$\dot{y}(t) = -b(y(t)) - u(t), \quad y(0) = x, \quad (8)$$

satisfies: $y(t) \xrightarrow{t \uparrow \infty} 0$.

We prove this in the subsequent sections.

3 The control formulation

As observed above

$$\int \mathcal{L}^\epsilon f d\mu^\epsilon = \int \mathcal{L}^\epsilon f(x) \varphi^\epsilon(x) dx = 0 \quad (9)$$

for smooth compactly supported f . Hence φ^ϵ satisfies

$$\frac{\epsilon^2}{2} \sum_{i,j=1}^n a_{ij} \varphi_{x_i x_j}^\epsilon + \sum_{i=1}^n \tilde{b}_i^\epsilon \varphi_{x_i}^\epsilon + c^\epsilon \varphi^\epsilon = 0, \quad (10)$$

or equivalently,

$$\frac{1}{2} \sum_{i,j=1}^n a_{ij} \varphi_{x_i x_j}^\epsilon + \sum_{i=1}^n \frac{\tilde{b}_i^\epsilon}{\epsilon^2} \varphi_{x_i}^\epsilon + \frac{c^\epsilon}{\epsilon^2} \varphi^\epsilon = 0, \quad (11)$$

where,

$$\begin{aligned} \tilde{b}_i^\epsilon &\stackrel{def}{=} -b_i + \epsilon^2 \sum_{j=1}^n (a_{ij})_{x_j}, \text{ for } i = 1, 2, \dots, n, \\ c^\epsilon &\stackrel{def}{=} \frac{\epsilon^2}{2} \sum_{i,j=1}^n (a_{ij})_{x_i x_j} - \sum_{i=1}^n (b_i)_{x_i}. \end{aligned}$$

Define

$$A^\epsilon \stackrel{def}{=} \frac{\epsilon^2}{2} \text{tr}(a \nabla^2) + \langle \tilde{b}^\epsilon, \nabla \rangle.$$

Letting $W^\epsilon(x) \stackrel{def}{=} -\epsilon^2 \ln(\varphi^\epsilon(x))$, W^ϵ is seen to satisfy

$$A^\epsilon W^\epsilon(x) - \frac{1}{2} DW^\epsilon(x)^T a(x) DW^\epsilon(x) - \epsilon^2 c^\epsilon(x) = 0, \quad (12)$$

or, equivalently,

$$\frac{\epsilon^2}{2} \sum_{i,j=1}^n a_{ij} W_{x_i x_j}^\epsilon + \min_{u \in \mathbb{R}^d} \left[(\tilde{b}^\epsilon - u)^T DW^\epsilon + \frac{1}{2} u^T a^{-1} u - \epsilon^2 c^\epsilon(x) \right] = 0. \quad (13)$$

Observe that this is the HJB equation for a certain ergodic control problem.

Lemma 1: $\bar{W}^\epsilon(\cdot) \stackrel{def}{=} W^\epsilon(\cdot) - W^\epsilon(0)$, $\epsilon \in (0, 1)$, is relatively compact in $C(\mathbb{R}^d)$.

Proof: In view of (10) and the Harnack estimates of Theorem 5.2 of [18], we have

$$\|D\bar{W}^\epsilon(x)\| = \|DW^\epsilon(x)\| \leq C \quad \forall x \in \mathbb{R}^d \quad (14)$$

for a constant C depending only on $\lambda, \|a\|_{C_b^4(\mathbb{R}^d)}, \|b\|_{C^2(\mathbb{R}^d)}$. Since $\bar{W}^\epsilon(0) = 0$, the claim follows by the Arzela-Ascoli theorem. \square

Now note that the minimum over u in (13) is attained at $u = a(x)DW^\epsilon(x)$, which is bounded by (14). Thus without loss of generality, for purposes of analysis we may a priori restrict this minimization to a closed bounded set Γ . Now letting $\epsilon \downarrow 0$, standard arguments from the theory of viscosity solutions ([3], Prop. VI.1) tell us that along an appropriate subsequence, $\bar{W}^\epsilon \rightarrow W$ uniformly on compacts, where W is a viscosity solution to the p.d.e.

$$\min_{u \in \Gamma} \left[(-b(x) - u)^T DW + \frac{1}{2} u^T a^{-1} u \right] = 0. \quad (15)$$

In view of (14), W is in fact Lipschitz. For the deterministic control system

$$\dot{x} = -b(x) - u(t), \quad (16)$$

define

$$\tilde{W}(x, t) = \inf_{x(0)=x} \left[\frac{1}{2} \int_0^t u(s)^T a^{-1}(x(s)) u(s) ds + W(x(t)) \right]$$

where the infimum is over all measurable and locally square-integrable $u(\cdot)$. Since W is Lipschitz, \tilde{W} will be a Lipschitz continuous viscosity solution [3] of

$$\begin{aligned} \tilde{W}_t(x, t) &= \inf_{u \in \Gamma} \left[(-b(x) - u)^T D\tilde{W}(x, t) + \frac{1}{2} u^T a^{-1}(x) u \right], \\ 0 < t < T, \quad \tilde{W}(x, 0) &= W(x). \end{aligned}$$

From Theorem VI.1 of [4], it follows that this equation has a unique viscosity solution and thus $W = \tilde{W}$. That is, W is a stationary solution to the above p.d.e, which is also its unique viscosity solution. In particular, W satisfies:

$$W(x) = \inf_{x(0)=x} \left[\frac{1}{2} \int_0^t u(s)^T a^{-1}(x(s)) u(s) ds + W(x(t)) \right], \quad (17)$$

where the infimum is over all $(x(\cdot), u(\cdot))$ satisfying (16) with $u(\cdot)$ locally square-integrable. We use this to establish several additional properties of W in the next section, leading to our main result.

4 Proof of Theorem 2

We proceed through a sequence of lemmas.

Lemma 2: $W(x) \geq 0$.

Proof: Suppose $\bar{W}^{\epsilon_n} \rightarrow W$ uniformly on compacts and suppose there exists $x_0 \neq 0$ such that $W(x_0) = -\delta$ for some $\delta > 0$. Let $B_r(x)$ denote the open ball of radius r centered at x . Then for a sufficiently small r , we have

$$\bar{W}^{\epsilon_n}(x) < -\frac{\delta}{2} \text{ for } x \in B_r(x_0); \quad \bar{W}^{\epsilon_n}(x) > -\frac{\delta}{4} \text{ for } x \in B_r(0).$$

So for n sufficiently large,

$$\begin{aligned} \frac{\mu^{\epsilon_n}(B_r(0))}{\mu^{\epsilon_n}(B_r(x_0))} &= \frac{\int_{B_r(0)} \varphi^{\epsilon_n}(x) dx}{\int_{B_r(x_0)} \varphi^{\epsilon_n}(x) dx} \\ &= \frac{\int_{B_r(0)} e^{-\frac{W^{\epsilon_n}(x)}{\epsilon_n^2}} dx}{\int_{B_r(x_0)} e^{-\frac{W^{\epsilon_n}(x)}{\epsilon_n^2}} dx} \\ &= \frac{\int_{B_r(0)} e^{-\frac{\bar{W}^{\epsilon_n}(x)}{\epsilon_n^2}} dx}{\int_{B_r(x_0)} e^{-\frac{\bar{W}^{\epsilon_n}(x)}{\epsilon_n^2}} dx} \\ &\leq \frac{|B_r(0)| e^{\frac{\delta}{4\epsilon_n^2}}}{|B_r(x_0)| e^{\frac{\delta}{2\epsilon_n^2}}} \rightarrow 0 \text{ as } n \uparrow \infty. \end{aligned}$$

This contradicts Theorem 1, proving the claim. \square

Lemma 3: For $\{\epsilon_n\}$ as above, $\lim_{n \uparrow \infty} W^{\epsilon_n}(0) = 0$.

Proof: We may write

$$\varphi^{\epsilon_n}(x) = \varphi^{\epsilon_n}(0) h_{\epsilon_n}(x) e^{-\frac{W(x)}{\epsilon_n^2}}$$

where $\epsilon_n^2 \ln(h_{\epsilon_n}(x)) \rightarrow 0$ uniformly on compact sets. Let $A \subset \mathbb{R}^d$ denote a compact set with 0 in its interior. Then

$$\epsilon_n^2 \ln \left(\int_A \varphi^{\epsilon_n}(x) dx \right) = \epsilon_n^2 \ln \left(\int_A \varphi^{\epsilon_n}(0) h_{\epsilon_n}(x) e^{-\frac{W(x)}{\epsilon_n^2}} dx \right).$$

We have

$$\left(\int_A e^{-\frac{W(x)}{\epsilon_n^2}} dx \right)^{\epsilon_n^2} \xrightarrow{n \uparrow \infty} \text{ess.sup} \{ e^{-W(x)} : x \in A \}$$

and

$$\begin{aligned}
& \epsilon_n^2 \ln(\varphi^{\epsilon_n}(0)) + \epsilon_n^2 \ln\left(\int_A e^{-\frac{W(x)}{\epsilon_n^2}} dx\right) + \epsilon_n^2 \ln\left(\inf_A h_{\epsilon_n}(x)\right) \\
& \leq \epsilon_n^2 \ln\left(\int_A \varphi^{\epsilon_n}(0) h_{\epsilon_n}(x) e^{-\frac{W(x)}{\epsilon_n^2}} dx\right) \\
& \leq \epsilon_n^2 \ln(\varphi^{\epsilon_n}(0)) + \epsilon_n^2 \ln\left(\int_A e^{-\frac{W(x)}{\epsilon_n^2}} dx\right) + \epsilon_n^2 \ln\left(\sup_A h_{\epsilon_n}(x)\right).
\end{aligned}$$

Therefore

$$-\epsilon_n^2 \ln\left(\int_A \varphi^{\epsilon_n}(x) dx\right) + \epsilon_n^2 \ln(\varphi^{\epsilon_n}(0)) \xrightarrow{n \uparrow \infty} \inf\{W(x) : x \in A\} = 0.$$

The first term on the left goes to zero by Theorem 1. Thus so does the second, proving the claim. \square

Lemma 4: $\lim_{\|x\| \rightarrow \infty} W(x) = \infty$.

Proof: Let $K > 0$, to be prescribed later and $\delta < \frac{2\alpha}{\beta}$ for α, β as in (2), $\theta, c > 0, m_1 \geq 2, m_2 \geq 4$. Define a C^2 function U^{ϵ_n} on \mathbb{R}^d satisfying:

$$\begin{aligned}
U^{\epsilon_n}(x) &= e^{\frac{\delta \|x\|^\beta}{\epsilon_n^2}} & \forall \|x\| > K, \\
&= 1 & \forall \|x\| \leq K - \theta \epsilon_n^2, \\
|\mathcal{L}^{\epsilon_n} U^{\epsilon_n}(x)| &< \frac{c}{(\theta \epsilon_n^2)^{m_1}} e^{\frac{m_2 \delta K^\beta}{\epsilon_n^2}} & \forall \|x\| \leq K.
\end{aligned}$$

Then for $\|x\| \geq K$,

$$\begin{aligned}
\mathcal{L}^{\epsilon_n} U^{\epsilon_n}(x) &= \delta \beta \|x\|^{\beta-1} e^{\frac{\delta \|x\|^\beta}{\epsilon_n^2}} \left[\frac{\sum_{i=1}^n a_{ii}}{2\|x\|} + \frac{\beta-1}{2\|x\|^3} \sum_{i,j=1}^n a_{ij} x_i x_j \right. \\
&\quad \left. + \frac{1}{\epsilon_n^2} \left(\frac{\delta \beta}{2} \|x\|^{\beta-3} \sum_{i,j=1}^n a_{ij} x_i x_j + b(x)^T \frac{x}{\|x\|} \right) \right].
\end{aligned}$$

In view of (2), we may choose K large enough such that the r.h.s. above is < 0 . Then by Proposition 2.3 of [18], $\int |\mathcal{L}^{\epsilon_n} U^{\epsilon_n}| d\mu^{\epsilon_n} < \infty$ and

$$\int_{\mathbb{R}^d} |\mathcal{L}^{\epsilon_n} U^{\epsilon_n}| d\mu^{\epsilon_n} \leq 2 \int_{B_K(0)} |\mathcal{L}^{\epsilon_n} U^{\epsilon_n}| d\mu^{\epsilon_n} \leq \frac{2c}{(\theta \epsilon_n^2)^{m_1}} e^{\frac{m_2 \delta K^\beta}{\epsilon_n^2}}.$$

Choose $K_1 > K$ such that

$$|\mathcal{L}^{\epsilon_n} U^{\epsilon_n}| \geq e^{\frac{\delta \|x\|^\beta}{2\epsilon_n^2}} \quad \forall \|x\| \geq K_1.$$

Then

$$\int_{B_R^c(0)} e^{\frac{\delta \|x\|^\beta}{2\epsilon_n^2}} d\mu^{\epsilon_n} \leq \frac{2c}{(\theta\epsilon_n^2)^{m_1}} e^{\frac{m_2\delta K^\beta}{\epsilon_n^2}}, \quad R > K_1..$$

Hence for $R > K_1$,

$$\mu^{\epsilon_n}(\bar{B}_R^c(0)) \leq \frac{2c}{(\theta\epsilon_n^2)^{m_1}} e^{\frac{2m_2\delta K^\beta - \delta R^\beta}{2\epsilon_n^2}}$$

Let $\bar{x} \in \bar{B}_R^c(0)$ and Θ an open and bounded neighborhood thereof. Then

$$\begin{aligned} -\epsilon_n^2 \ln([\inf_{x \in \Theta} \varphi^{\epsilon_n}(x)]|\Theta|) &\geq -\epsilon_n^2 \ln\left(\int_{\bar{B}_R(0)} \varphi^{\epsilon_n}(x) dx\right) \\ &\geq -\epsilon_n^2 \ln(2c) + m_1 \epsilon_n^2 \ln(\theta\epsilon_n^2) + \frac{\delta R^\beta}{2} - m_2 \delta K^\beta, \end{aligned}$$

leading to, in view of Lemma 3,

$$\sup_{\Theta} W(x) \geq \frac{\delta R^\beta}{2} - m_2 \delta K^\beta.$$

For R large enough and Θ small enough, we have by the Lipschitz continuity of W that $W(\bar{x}) \geq \frac{\delta R^\beta}{4}$, which implies the result in view of our arbitrary choice of \bar{x} . \square

Lemma 5: $W(x) > 0$ for $x \neq 0$.

Proof: Suppose $W(x) = 0$ for some $x \neq 0$. Considering $x(0) = x$ in (16), we have

$$0 = W(x) = \inf \left[\frac{1}{2} \int_0^T u(t)^T a(x(t))^{-1} u(t) dt + W(x(T)) \right] \quad \forall T > 0,$$

implying that the infimum is in fact attained at $u(\cdot) \equiv 0$ and the corresponding $x(\cdot)$ satisfies $W(x(T)) = 0 \quad \forall T > 0$. But in view of (16) with $u(\cdot) \equiv 0$ and our hypotheses on $b(\cdot)$, $x(T) \uparrow \infty$, leading to $W(x(T)) \uparrow \infty$, a contradiction. The result follows. \square

Thus we have:

Lemma 6: $W(x) = \inf \left[\frac{1}{2} \int_0^\infty u(t)^T a^{-1}(x(t)) u(t) dt \right]$ where the infimum is over all $(x(\cdot), u(\cdot))$ satisfying (16) with the additional restrictions: $x(0) = x$ and $x(t) \xrightarrow{t \uparrow \infty} 0$.

Proof: We topologize the space of locally square-integrable $u(\cdot)$ as follows: For $T > 0$, let $L_w^2[0, T]$ denote the space $L^2([0, T]; \mathbb{R}^d)$ with weak* topology. Equip the set of admissible $u(\cdot)$ with the coarsest topology that renders continuous the map $u(\cdot) \rightarrow u(\cdot)|_{[0, T]} \in L_w^2[0, T]$ for all $T > 0$. Now consider the minimization problem of minimizing over $u(\cdot) \in L_w^2[0, T]$ the functional

$$\frac{1}{2} \int_0^T u(t)^T a(x(t))^{-1} u(t) dt + W(x(T)), \quad (18)$$

where $x(\cdot), u(\cdot)$ are related through (16), with $x(0) = x$ (say). Let

$$\mathcal{U}_T = \{u \in L_w^2[0, T] \mid u(t) \in \Gamma \text{ a.e.}\}$$

It is easy to verify that (18) is a lower semicontinuous functional which will attain its minimum over a compact set \mathcal{U}_T of $L_w^2[0, T]$. Let $T' > T$. Then a standard dynamic programming argument shows that

$$u(\cdot) \in \mathcal{U}_{T'} \implies u(\cdot)|_{[0, T]} \in \mathcal{U}_T.$$

Define $\mathcal{U}_T^* \stackrel{def}{=} \{u(\cdot) \in \mathcal{U} (\equiv \mathcal{U}_\infty) \text{ such that } u(\cdot)|_{[0, T]} \in \mathcal{U}_T\}$. Then as $T \uparrow \infty$, it is a family of nested decreasing compact subsets of \mathcal{U} and therefore has a non-empty intersection. Take $u^*(\cdot)$ in this intersection and let $x^*(\cdot)$ denote the corresponding trajectory of (16). Then

$$W(x) = \frac{1}{2} \int_0^T u^*(t)^T a(x^*(t))^{-1} u^*(t) dt + W(x^*(T)) \quad \forall T,$$

implying that $W(x^*(T)) \downarrow W^*$ for some $W^* \geq 0$. Suppose $W^* > 0$. Then there exist $0 < \delta < R$ such that $\delta < x^*(s) \leq R \quad \forall s$. By Lemma 3.1 of [9], we then have

$$W(x) \geq \frac{1}{2} \int_0^T u^*(t)^T a^{-1}(x^*(t)) u^*(t) dt \rightarrow \infty,$$

a contradiction. Hence $W^* = 0$. By Lemma 5, we have $x^*(T) \xrightarrow{T \uparrow \infty} 0$. The rest is easy. \square

Proof of Theorem 2: Immediate from the above lemmas. \square

Let $R_\epsilon = \epsilon^d e^{W/\epsilon^2} \varphi^\epsilon$. In [6] it was shown that for a certain class of drifts, W is C^1 in an open, dense, connected set $\mathbf{G} \subset \mathbb{R}^d$ and

$$R_\epsilon \rightarrow R_0 \quad \text{uniformly on compact subset in } \mathbf{G},$$

where R_0 satisfies the equation

$$\langle b + a \nabla W, \nabla R_0 \rangle + (\operatorname{div}(W) + \frac{1}{2} \Delta W) R_0 = 0 \quad (19)$$

in \mathbf{G} . The same result can be generalized for the class of drifts considered here, which subsume the drifts considered in [6] for $\beta < 1$. This is because the asymptotic stability of 0 for (3) implies that the Jacobian matrix $Db(0)$ will have eigenvalues with strictly negative real parts (see, e.g., section 1.3 of [15]). We state it as theorem below and omit the proof which is identical to that of Theorem 3 [6]. (Of course, the equation (19) gets suitably modified.)

Theorem 3: There exists a positive function $R_0 \in C^1(\mathbf{G})$ such that $R_\epsilon \xrightarrow{\epsilon \downarrow 0} R_0$.

5 Extension to multiple equilibria

Now consider the case when (3) has finitely many equilibria x_1, \dots, x_J (say), and no other ω -limit sets. In this case, one can mimic the arguments of Theorem 1 to claim that all limit points of μ^ϵ as $\epsilon \downarrow 0$ concentrate on the set $\{x_1, \dots, x_J\}$. Furthermore, we again have W as the Lipschitz continuous subsequential limit of $-\epsilon^2 \ln \left(\frac{\varphi^\epsilon(x)}{\varphi^\epsilon(0)} \right)$ as $\epsilon \downarrow 0$, which satisfies (17). Also, by arguments of Lemma 4, we have $\lim_{\|x\| \uparrow \infty} W(x) = \infty$. The proof of Lemma 2 can be adapted to show that the limit points of μ^ϵ as $\epsilon \downarrow 0$ will in fact concentrate on the set $\operatorname{Argmin}(W(x) : x \in \{x_1, \dots, x_J\})$.

Let $(\hat{x}(\cdot), \hat{u}(\cdot))$ denote an optimal pair with $\hat{x}(0) = x$, as in Lemma 6. Then for $t, T > 0$,

$$W(\hat{x}(t)) = \frac{1}{2} \int_t^{t+T} \hat{u}(s)^T a^{-1}(\hat{x}(s)) \hat{u}(s) ds + W(\hat{x}(t+T)). \quad (20)$$

Thus $W(\hat{x}(t))$ is non-increasing with t , implying in particular that $\hat{x}(\cdot)$ remains bounded. Thus $\int_t^{t+T} \hat{u}(s)^T a^{-1}(\hat{x}(s)) \hat{u}(s) ds$ is bounded. By Banach-Alaoglu theorem, $\hat{u}(t + \cdot)|_{[0, T]}, t \geq 0$, is relatively compact in $L_w^2[0, T]$. By a standard argument using Arzela-Ascoli theorem, one verifies that $\hat{x}(t + \cdot)|_{[0, T]}$ is also relatively compact in $C([0, T]; \mathbb{R}^d)$. Let $(x^*(\cdot), u^*(\cdot))$ denote a subsequential limit of $(\hat{x}(t + \cdot), \hat{u}(t + \cdot))$ in $C([0, T]; \mathbb{R}^d) \times L_w^2[0, T]$ as $t \uparrow \infty$. Then by the lower semicontinuity of the map

$$\hat{u}(\cdot) \in L_w^2[0, T] \rightarrow \int_0^T \hat{u}(s)^T a^{-1}(\hat{x}(s)) \hat{u}(s) ds,$$

we have

$$W(x^*(0)) \geq \frac{1}{2} \int_0^T u^*(s)^T a^{-1}(x^*(s)) u^*(s) ds + W(x^*(T)). \quad (21)$$

But $W(\hat{x}(t))$ is monotonically decreasing by virtue of (20), whence $W(x^*(0)) = W(x^*(T)) = c \stackrel{\text{def}}{=} \lim_{t \uparrow \infty} W(\hat{x}(t))$. Thus $u^*(\cdot) \equiv 0$ a.e. We drop the ‘a.e.’ without loss of generality. It follows that $\hat{x}(t + \cdot)$ converges as $t \uparrow \infty$ to $H = \cap_{t > 0} \overline{\{\hat{x}(t + s), s \geq 0\}}$, which is a positively invariant set of the o.d.e.

$$\dot{x}(t) = -b(x(t)).$$

A similar argument can be given to establish negative invariance of this set as well, implying that it converges to an invariant set of the above o.d.e. But only such sets are the equilibria of (3). Since H is compact connected (being intersection of such), it must converge to a single equilibrium. Letting $t = 0$ and $T \uparrow \infty$ in (20), we then have

$$W(x) = \frac{1}{2} \int_0^\infty \hat{u}(t)^T a^{-1}(\hat{x}(t)) \hat{u}(t) dt + W(x_i)$$

for some $1 \leq i \leq J$. Since we also have

$$W(x) = \inf \left[\frac{1}{2} \int_0^T u(t)^T a^{-1}(x(t)) u(t) dt + W(x(T)) \right],$$

it follows by a straightforward argument that:

Theorem 4:

$$W(x) = \min_{1 \leq i \leq J} \inf_{(x(\cdot), u(\cdot)): x(t) \rightarrow x_i} \left[\frac{1}{2} \int_0^\infty u(t)^T a^{-1}(x(t)) u(t) dt + W(x_i) \right]. \quad (22)$$

In particular, it follows that W attains its minimum at one or more of the x_i ’s. As W needs be specified only up to an additive factor, we may assume without loss of generality that its minimum value is zero.

Note, however, that unlike in the single equilibrium case, the uniqueness of W , obtained as a *subsequential* limit, is not immediate.

Theorem 5: W is uniquely specified as the Lipschitz function satisfying (22) and the condition $\min W = 0$.

Proof: Note that by (22), we have by arguments similar to those used in Lemma 6 that for each x , there is an optimal pair $(x^*(\cdot), u^*(\cdot))$ such that $x^*(t) \rightarrow x_{i^*}$ for some i^* that will depend on x , and

$$W(x) = \frac{1}{2} \int_0^\infty u^*(t)^T a^{-1}(x^*(t)) u^*(t) dt + W(x_{i^*}).$$

If there is more than one i^* for which this holds, we choose one according to some pre-specified rule. In this case, write $x \rightarrow x_{i^*}$. We may also have $x_i \rightarrow x_j$ for some $i \neq j$, with $W(x_j) < W(x_i)$. In this case, write $x_i \Rightarrow x_j$. If we draw a directed graph with nodes $\{x_1, \dots, x_J\}$ with a directed edge from x_i to x_j whenever $x_i \Rightarrow x_j$, we get a forest of rooted trees, say, T_1, \dots, T_K . For each T_i , let \hat{x}_i denote its root. Let $O_i \stackrel{def}{=} \{x : x \rightarrow x_j \text{ for some } x_j \in T_i\}$. Then \mathbb{R}^d is the disjoint union of the O_i 's. On each O_i , W is completely specified in terms of $W(\hat{x}_i)$ by successive application of (22) as follows: First use (22) to obtain values of W at nodes $\{x_j\}$ in T_i one removed from \hat{x}_i , then repeat the same for nodes two removed, and so on till W is defined for all nodes in T_i . Then define it for $x \in O_i$ by using (22) again. In particular, W is completely specified for those T_i for which $W(\hat{x}_i) = 0$. For others, it is in principle specified up to an additive scalar, because the value of $W(\hat{x}_i)$ is not specified. But we have the additional restriction that W be continuous (in fact, Lipschitz) over the whole of \mathbb{R}^d , whence this choice is also unique. Uniqueness of W as $\lim_{\epsilon \downarrow 0} \left(-\epsilon^2 \ln(\varphi^\epsilon(\cdot)) \right)$ follows. \square

Remark: The problem of finding stationary density is one of finding the principal eigenfunction of an operator. A counterpart of the above in this general framework, albeit in a discrete set-up, appears in [22].

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Erratum for ‘Small noise asymptotics for invariant densities for a class of diffusions: a control theoretic view’

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The uniqueness argument in the proof of Theorem 5, p. 483, of [1] is flawed. We give here a corrected proof.

Define

$$S_{0T}(\phi) = \frac{1}{2} \int_0^T (\dot{\phi}(s) - b(\phi(s)))a^{-1}(\phi(s))(\dot{\phi} - b(\phi(s)))ds.$$

and

$$V_i(x) = \inf\{S_{0T}(\phi) : \phi(0) = x_i, \phi(T) = x\},$$

where the quantities are defined for absolutely continuous paths.

Lemma 0.1 *Under the stated assumption in [1], $V_i(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$ for $i \leq i \leq J$.*

Proof. Consider the stochastic differential equation

$$dY^\epsilon(t) = b(Y^\epsilon(t))dt + \epsilon dB(t),$$

where $B(\cdot)$ is a d -dimensional Brownian motion. Then following the arguments in [1], page 477–479, we have a Lipschitz continuous viscosity solution $U(\cdot)$ to the p.d.e.

$$\min_{u \in \mathbb{R}^d} [(-b(x) - u) \cdot \nabla U + \frac{1}{2} \|u\|^2] = 0.$$

Also arguing as in Lemma 2, [1] we have $U(\cdot) \geq 0$. As in [1], we define

$$\tilde{U}(x, t) = \inf_{x(0)=x} [\frac{1}{2} \int_0^t \|u(s)\|^2 ds + U(x(t))],$$

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where the infimum is taken over all locally square integrable $u(\cdot)$ satisfying $\dot{x}(s) = -b(x(s)) - u(s)$, $x(0) = x$. One can easily check that $\tilde{U}(x, t)$ is continuous on $\mathbb{R}^d \times [0, T]$ and Lipschitz in space variable uniformly w.r.t. $t \in [0, T]$. Also $\tilde{U}(x, t)$ is a viscosity solution to the equation

$$\tilde{U}_t(x, t) = \min_{u \in \mathbb{R}^d} [(-b(x) - u) \cdot \nabla \tilde{U} + \frac{1}{2} \|u\|^2].$$

Therefore applying Theorem VII.2. in [2] (see also the Appendix) we have

$$U(x) = \inf_{x(0)=x} [\frac{1}{2} \int_0^T \|u(s)\|^2 ds + U(x(T))], \quad (0.1)$$

where the infimum is over all $(x(\cdot), u(\cdot))$ satisfying

$$\dot{x}(t) = -b(x(t)) - u(t), \quad x(0) = x$$

with $u(\cdot)$ locally square integrable. Furthermore, applying Lemma 4 in [1] we have $U(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. Suppose there exists a sequence $\{y_n\}$ such that $\|y_n\| \uparrow \infty$ and $V_i(y_n) \leq M$ for all $n \geq 1$ and some $0 < M < \infty$. Therefore there exists $\{\phi_n(\cdot)\}$ such that

$$S_{0T_n}(\phi_n) \leq M + 1, \quad \phi_n(0) = x_i, \quad \phi_n(T_n) = y_n.$$

Define $\psi_n(s) = \phi_n(T_n - s)$ for $s \leq T_n$ and

$$u_n(s) = \begin{cases} -\dot{\psi}_n(s) - b(\psi_n(s)) & \text{for } s \leq T_n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\dot{\psi}_n(s) = -b(\psi_n(s)) - u(s)$. Hence from (0.1) we have

$$\begin{aligned} U(y_n) &\leq \frac{1}{2} \int_0^{T_n} \|u_n(s)\|^2 ds + U(x_i) \\ &= \Lambda S_{0T_n}(\phi_n) + U(x_i) \leq \Lambda(M + 1) + U(x_i), \end{aligned}$$

where Λ is the maximal eigenvalue of $a(\cdot)$. This contradicts the fact that $U(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. This proves the claim. \square

Since there are only finitely many equilibria, it is easy to see that $V(x) := \min_{1 \leq i \leq J} V_i(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. We introduce the following notations:

$$\tilde{V}(x_i, x_j) = \inf \{S_{0T}(\phi) : \phi(0) = x_i, \phi(T) = x_j, \phi(s) \in \mathbb{R}^d \setminus \cup_{k \neq i, j} \{x_k\}\}.$$

Let B_R denotes the ball of radius R around 0. For R large, define

$$\begin{aligned} \tilde{V}_R(x_i, x_j) &= \inf \{S_{0T}(\phi) : \phi(0) = x_i, \phi(T) = x_j, \\ &\quad \phi(s) \in \bar{B}_R \setminus \cup_{k \neq i, j} \{x_k\} \quad \forall s \in [0, T]\}. \end{aligned}$$

Let ρ_1, ρ_0 be such that $0 < \rho_1 < \rho_0 < \min_{1 \leq i, j \leq J} \|x_i - x_j\|$. Let Γ_i, g_i denote the boundaries of the balls of radius ρ_0, ρ_1 respectively around x_i . Denote $\Gamma = \cup_{1 \leq i \leq J} \Gamma_i$ and $g = \cup_{1 \leq i \leq J} g_i$. Define $\tau_0 = 0, \sigma_n = \inf\{t \geq \tau_n : X^\epsilon(t) \in \Gamma\}, \tau_n = \min\{t \geq \sigma_{n-1} : X^\epsilon(t) \in g\}$, where $X^\epsilon(\cdot)$ satisfies the stochastic differential equation (1) in [1], i.e.,

$$dX^\epsilon(t) = b(X^\epsilon(t))dt + \epsilon\sigma(X^\epsilon(t))dB(t).$$

Define the Markov chain $Z_n := X^\epsilon(\tau_n)$.

Lemma 0.2 *For any $\gamma > 0$ there exists ρ_0 small such that for any $\rho_2, 0 < \rho_2 < \rho_0$, there exists $\rho_1, 0 < \rho_1 < \rho_2$, such that for ϵ sufficiently small, for all $x \in g_i$ the one-step transition probabilities of Z_n satisfy the inequalities*

$$\exp\left(-\frac{\tilde{V}(x_i, x_j) + \gamma}{\epsilon^2}\right) \leq P(x, g_j) \leq \exp\left(-\frac{\tilde{V}(x_i, x_j) - \gamma}{\epsilon^2}\right).$$

Proof. We choose $R > 0$ large enough such that following holds:

- Any solution of $\dot{x}(t) = b(x(t))$ that starts in \bar{B}_R , stays in \bar{B}_R ,
- $\tilde{V}_R(x_i, x_j) - \frac{\gamma}{2} \leq \tilde{V}(x_i, x_j) \leq \tilde{V}_R(x_i, x_j)$ for all $1 \leq i, j \leq J$,
- $\min_{x \in \partial B_R} V(x) > \tilde{V}(x_i, x_j) + 4\gamma$ for all $1 \leq i, j \leq J$.

Therefore applying Lemma 6.2.1 in [3] we have for $x \in g_i$,

$$P(x, g_j) \geq \exp\left(-\frac{\tilde{V}_R(x_i, x_j) + \gamma/2}{\epsilon^2}\right) \geq \exp\left(-\frac{\tilde{V}(x_i, x_j) + \gamma}{\epsilon^2}\right)$$

for ϵ small. To get the upper bound we follow the calculations in ([3], page 174). Following the calculations on page 174 (estimate (2.6)) of [3] it is enough to bound $P_x(\tau_1 > T)$ by $\exp\left(-\frac{\tilde{V}(x_i, x_j) - \gamma/2}{\epsilon^2}\right)$ for a chosen $T > 0$ and $x \in \Gamma_i$. At this point we define $\xi = \min\{t > 0 : X^\epsilon(t) \in \partial B_R\}$. Let $V_0 = \min_{x \in \partial B_R} V(x)$. Now for $x \in \Gamma_i$

$$P_x(\tau_1 > T) \leq P_x(\tau_1 > T, \xi > T) + P_x(\xi \leq T).$$

Using Lemma 6.1.9 in [3], we can choose T large such that $\sup_{x \in \Gamma_i} P_x(\tau_1 > T, \xi > T) \leq \exp\left(-\frac{V_0}{\epsilon^2}\right)$ for ϵ small enough. We fix this choice of T . To estimate the other term we define $\Phi(x) = \{\phi : S_{0T}(\phi) \leq V_0 - \gamma/4\}$ for $x \in \Gamma_i$. Now any path starting at x that leaves B_R by time T must be at least η distance away in $C([0, T], \mathbb{R}^d)$ from $\Phi(x)$ for ρ_0, η small and $x \in \Gamma_i$. Therefore using the Freidlin-Wentzell large deviations principle we have (see also [3], page 174)

$$\sup_{x \in \Gamma_i} P_x(\xi \leq T) \leq \sup_{x \in \Gamma_i} P_x(\text{dist}_{0T}(X^\epsilon, \Phi(x)) \geq \eta) \leq \exp\left(-\frac{V_0 - \gamma/2}{\epsilon^2}\right)$$

for ϵ small where $\text{dist}_{0T}(\phi_1, \phi_2) = \sup_{[0,T]} \|\phi_1(s) - \phi_2(s)\|$ is the uniform metric on $C([0, T], \mathbb{R}^d)$. This completes the proof. \square

Taking $\mathcal{V} = \|x\|^2$, for ϵ small we have from [1] (estimate (4) of [1])

$$\mathcal{L}^\epsilon \mathcal{V} \leq -\delta_2, \quad \|x\| \geq K. \quad (0.2)$$

for some positive δ_2 . Let $D_1 = B_K$ and $D = B_{K+1}$. We need to estimate the hitting time $\tau = \min\{t : X^\epsilon(t) \in \cup_i \partial g_i\}$. Let O_i denote the closed ball of radius ρ_1 around x_i .

Lemma 0.3 *Let $\mathbb{K} = \overline{D_1 \setminus \cup O_i}$. Then there exists $M_1 > 0$ such that*

$$\sup_{x \in \mathbb{K}} E_x[\tau] \leq M_1$$

for all $\epsilon \in (0, \epsilon_0)$.

Proof. Define $\tau_0 = \min\{t : X^\epsilon(t) \notin D \setminus \cup_i O_i\}$, $\sigma_i = \min\{t \geq \tau_{i-1} : X^\epsilon(t) \in \partial D_1\}$, $\tau_i = \min\{t \geq \sigma_i : X^\epsilon(t) \notin D \setminus \cup_i O_i\}$. Using (0.2) we can show that

$$\sup_{x \in \partial D} E_x[\sigma_1] \leq \sup_{x \in \partial D} \frac{\mathcal{V}(x)}{\delta_2} = M_2.$$

Also, Lemma 1.9, pp. 168, of [3] shows that

$$\sup_{\mathbb{K}} E_x[\tau_0] \leq M_3,$$

for $\epsilon > 0$ small. If for some i , $X^\epsilon(\tau_i) \in \cup_i \partial g_i$ then we define $\sigma_{i+1} = \tau_{i+1} = \dots = \tau_i$. By positive recurrence we know that $P(\tau < \infty) = 1$. Hence for $x \in \mathbb{K}$

$$\begin{aligned} E_x[\tau] &= E_x\left[\sum_{i=0}^{\infty} \tau_i \mathbb{I}_{\{\tau=\tau_i, \tau > \tau_{i-1}\}}\right] \\ &= E_x\left[\sum_{i=0}^{\infty} \sum_{k=0}^i (\tau_k - \tau_{k-1}) \mathbb{I}_{\{\tau=\tau_i, \tau > \tau_{i-1}\}}\right] \text{ [with } \tau_{-1} = 0] \\ &= E_x\left[\sum_{k=0}^{\infty} \sum_{i=k}^{\infty} (\tau_k - \tau_{k-1}) \mathbb{I}_{\{\tau=\tau_i, \tau > \tau_{i-1}\}}\right] \\ &= E_x\left[\sum_{k=0}^{\infty} (\tau_k - \tau_{k-1}) \mathbb{I}_{\{\tau > \tau_{k-1}\}}\right] \\ &\leq \sup_{x \in \mathbb{K}} E_x[\tau_0] + E_x\left[\sum_{k=1}^{\infty} (\tau_k - \tau_{k-1}) \mathbb{I}_{\{\tau > \tau_{k-1}\}}\right] \\ &= M_3 + \sum_{k=1}^{\infty} E_x[(\tau_k - \tau_{k-1}) \mathbb{I}_{\{\tau > \tau_{k-1}\}}] \\ &\leq M_3 + \sup_{x \in \partial D} E_x[\tau_1] \sum_{k=1}^{\infty} P(\tau > \tau_{k-1}) \end{aligned}$$

For $x \in \partial D$ we observe that $E_x[\tau_1] = E_x[\tau_1 - \sigma_1 + \sigma_1] \leq M_3 + M_2$. Also for $k = 1$, $\sup_{x \in \mathbb{K}} P(\tau > \tau_0) < \delta$ for some $\delta < 1/2$ and $\epsilon > 0$ small. (This is because then the probability concentrates around the deterministic trajectories of the o.d.e. for $\epsilon = 0$, which always hit $\cup_i \partial g_i$ before hitting ∂D .) So for $k \geq 2$,

$$\begin{aligned} P(\tau > \tau_{k-1}) &\leq P(\tau > \sigma_{k-1}) \sup_{x \in \partial D_1} P_x(\tau > \tau_0) \\ &\leq \delta P(\tau > \tau_{k-2}) \\ &\leq \delta^{k-1}. \end{aligned}$$

Hence combining with the above calculations we have $\sup_{x \in \mathbb{K}} E_x[\tau] \leq M_1 < \infty$ for all $\epsilon \in (0, \epsilon_0)$. \square

Let $G(i)$ denote the set of all i -graphs defined on page 177 in [3]. Define

$$Z(x_i) = \min_{\chi \in G(i)} \sum_{(m \rightarrow n) \in \chi} \tilde{V}(x_m, x_n).$$

Recall that μ^ϵ is the invariant measure of $X^\epsilon(\cdot)$. Proof of the following theorem is same as the proof of Theorem 6.4.1 in [3]

Theorem 0.1 *For any $\gamma > 0$ there exists $\rho_1 > 0$ (can be chosen arbitrary small) such that the μ^ϵ -measure of $x_i + B_{\rho_1}$ is between*

$$\exp\left(-\frac{Z(x_i) - \min_j Z(x_j) \pm \gamma}{\epsilon^2}\right)$$

for ϵ sufficiently small.

Now we are ready to prove Theorem 5 in [1].

Theorem 0.2 *The function $W(\cdot)$ obtained in Theorem 4 of [1] is unique and given by*

$$W(x_i) = \min_{1 \leq i \leq J} \inf_{(x(\cdot), u(\cdot)): x(t) \rightarrow x_i} \left[\frac{1}{2} \int_0^\infty u(t)^T a^{-1}(x(t)) u(t) dt + W(x_i) \right],$$

where $(x(\cdot), u(\cdot))$ satisfies (16) in [1].

Proof. The above expression implies that $W(\cdot)$ is uniquely determined by the values $W(x_i), 1 \leq i \leq J$. Recall that there exists a subsequence $\{\epsilon_n\}$ such that $-\epsilon_n^2 \ln(\varphi^{\epsilon_n}(x)) \rightarrow W(x)$ uniformly on compact subsets of \mathbb{R}^d as $\epsilon_n \rightarrow 0$ where $\varphi^\epsilon(\cdot)$ denotes the invariant density of μ^ϵ . To prove the uniqueness it is enough to show that the values of $W(x_i), 1 \leq i \leq J$, are uniquely determined. Let $\gamma > 0$ be small. Choose ρ_1 small enough so that

$|W(x) - W(x_i)| < \gamma$ for all $\|x - x_i\| \leq \rho_1$. Therefore applying Theorem 0.1 we have

$$-\epsilon_n^2 \ln \left[\sup_{B_{\rho_1}(x_i)} \varphi^{\epsilon_n} |B_{\rho_1}(x_i)| \right] \leq Z(x_i) - \min_j Z(x_j) + \gamma,$$

for n large enough, where $|A|$ denotes the Lebesgue measure of A . Therefore letting $n \rightarrow \infty$ we have

$$\inf_{B_{\rho_1}(x_i)} W(x) \leq Z(x_i) - \min_j Z(x_j) + \gamma,$$

and hence

$$W(x_i) \leq Z(x_i) - \min_j Z(x_j) + 2\gamma.$$

Similarly using the upper bound in Theorem 0.1 we have

$$W(x_i) \geq Z(x_i) - \min_j Z(x_j) - 2\gamma.$$

Since $\gamma > 0$ is arbitrary, we have $W(x_i) = Z(x_i) - \min_j Z(x_j)$. This completes the proof of the theorem. \square

Other minor corrections

- The closed bounded set Γ in the expression (15), page 479 in [1], should be taken as a closed bounded ball with center at the origin.
- We need to assume that $\frac{\partial a_{ij}}{\partial x}(0) = 0$, for all i, j , in Theorem 3 in [1].

Appendix

Here we prove uniqueness of viscosity solution of the Hamilton Jacobi equation

$$\begin{aligned} u_t + H(x, Du) &= 0 \text{ in } \mathbb{R}^d \times (0, T] \\ u(x, 0) &= g(x). \end{aligned} \tag{0.3}$$

We prove the uniqueness along the lines of [2]. We assume the following

(A1) For some local modulus $\sigma : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$

$$H(y, \lambda(x - y)) - H(x, \lambda(x - y)) \leq \sigma(\lambda\|x - y\|^2 + \|x - y\|, R)$$

for $R > 0, x, y \in B_R, \lambda \geq 0$.

(A2) There exists a constant $C_0 > 0$ such that

$$|H(x, p) - H(x, q)| \leq C_0(1 + \|p\| + \|q\|)\|p - q\|$$

for all $x, p, q \in \mathbb{R}^d$.

Let u, v be two viscosity solution of (0.3) which are continuous on $\mathbb{R}^d \times [0, T]$ and Lipschitz in space variable uniformly w.r.t. $t \in [0, T]$. To show that $u = v$ on $\mathbb{R}^d \times [0, T]$ it is enough to show that

$$\sup_{\mathbb{R}^d \times [0, T]} (u - v)^+ \leq 0.$$

Now if possible, let there exists $\chi > 0$ such that for some $t_0 > 0$, $x_0 \in \mathbb{R}^d$,

$$u(x_0, t_0) - v(x_0, t_0) \geq 3\chi.$$

Let

$$|u(x, t) - u(y, t)| \leq K\|x - y\| \text{ and } |v(x, t) - v(y, t)| \leq K\|x - y\|$$

for all $t \in [0, T]$. Therefore from assumption (A2) we have

$$|H(x, p) - H(x, q)| \leq C\|p - q\| \quad (0.4)$$

for $p \in D_x^+ u(x, t)$, $q \in D_y^- v(y, s)$ for all $x, y \in \mathbb{R}^d$ and $s, t \in (0, T]$. For $0 < \theta < 1$, define

$$w_\delta(x, t) = \exp \left\{ \frac{1}{\delta} \left(\log(1 + \|x\|^2) + \frac{CT^\theta}{1-\theta} t^{1-\theta} - R \right) \right\}.$$

It is easy to check that

$$\frac{\partial w_\delta}{\partial t} - C \frac{T^\theta}{t^\theta} \|Dw_\delta\| \geq 0 \text{ in } \mathbb{R}^d \times (0, \infty).$$

Choose R large enough so that $\log(1 + \|x_0\|^2) + \frac{CT^\theta}{1-\theta} t_0^{1-\theta} - R < 0$. Now define $Q_R = \{\delta \log w_\delta(x, t) \leq 1\} \subset \mathbb{R}^d \times [0, T]$. It is easy to see that there exists a compact set \mathbb{K} independent of δ such that $Q_R \subset \mathbb{K} \times [0, T]$ for all $\delta > 0$. Let $M = (\max_{Q_R} u(x, t) \vee \max_{Q_R} v(x, t))$. Now we choose δ small so that $w_\delta(x_0, t_0) < (\chi \wedge 1)$ and $e^{\frac{1}{2\delta}} > 2M$. On $Q_R \times Q_R$, define

$$\Phi(x, t, y, s) = u(x, t) - v(y, s) - w_\delta(x, t) - \frac{\|x - y\|^2 + |t - s|^2}{\epsilon} - a(t + s)$$

for $a, \epsilon > 0$. For $a > 0$ small, we have

$$\sup_{Q_R \times Q_R} \Phi \geq \Phi(x_0, t_0, x_0, t_0) \geq \chi > 0.$$

Let $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$ be the point where the maximum of Φ is attained. By standard calculations, it can be shown that

$$\bullet \quad \frac{\|\bar{x} - \bar{y}\|^2 + |\bar{t} - \bar{s}|^2}{\epsilon} \leq 2M.$$

- $\frac{\|\bar{x}-\bar{y}\|^2}{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Using the uniform continuity of u, v, w_δ on Q_R and by our choice of δ it is easy to show that $\bar{t} > 0, \bar{s} > 0$ for $\epsilon > 0$ small. Also for the same reasons, $\delta \log w_\delta(\bar{x}, \bar{t}) < 1$ and $\delta \log w_\delta(\bar{y}, \bar{s}) < 1$. Hence we can apply the definition of viscosity solution to the points $(\bar{x}, \bar{t}), (\bar{y}, \bar{s})$ to get

$$a + \frac{\partial w_\delta(\bar{x}, \bar{t})}{\partial t} + 2\frac{\bar{t} - \bar{s}}{\epsilon} + H\left(\bar{x}, Dw_\delta(\bar{x}, \bar{t}) + 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) \leq 0,$$

and

$$-a + 2\frac{\bar{t} - \bar{s}}{\epsilon} + H\left(\bar{y}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) \geq 0.$$

Subtracting we have

$$\begin{aligned} 2a + \frac{\partial w_\delta(\bar{x}, \bar{t})}{\partial t} + H\left(\bar{x}, Dw_\delta(\bar{x}, \bar{t}) + 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) - H\left(\bar{x}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) \\ + H\left(\bar{x}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) - H\left(\bar{y}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) \leq 0 \end{aligned}$$

Therefore using (0.4) we have

$$2a + \frac{\partial w_\delta(\bar{x}, \bar{t})}{\partial t} - \frac{CT^\theta}{\bar{t}^\theta} |Dw_\delta(\bar{x}, \bar{t})| + H\left(\bar{x}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) - H\left(\bar{y}, 2\frac{\bar{x} - \bar{y}}{\epsilon}\right) \leq 0$$

Now letting $\epsilon \rightarrow 0$ and using (A1) we have $2a \leq 0$ which is a contradiction. This proves the uniqueness claim.

References

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